NOTE ON A SUBSTITUTE FOR DUHAMEL'S THEOREM.

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The substitute for Duhamel's Theorem published by G. A. Bliss in the sixteenth volume of this journal* meets all the requirements of the case for students who are mature enough mathematically to understand it. But it is cast in too abstract a form to be serviceable as early in the study of the calculus as some theorem of this kind must be used. A less general theorem is all that is needed, but both in statement and proof it must be sufficiently elementary to be intelligible to a student soon after he has become familiar with the notion of the simple definite integral as the limit of a sum of infinitesimal elements and has acquired some skill in applying this notion. I am therefore led to suggest the following theorem. In content it is the equivalent of a theorem recently published by E. V. Huntington† and like that is a particular case of Bliss's theorem, but the proof, unlike Huntington's and like Bliss's, shows directly the existence of the limit of the sum in question.

THEOREM. *Let $f_1(x), f_2(x), \cdots, f_p(x)$ denote any set of functions of x, finite in number, which are continuous in the interval (ab), and let*

$$F(x) = f_1(x)f_2(x)\cdots f_p(x).$$

Suppose the interval (ab) to be divided and redivided into parts in any manner such that as the process is indefinitely continued the greatest of the parts will approach 0 as limit, and at any stage in the process let $h_1, h_2, \cdots, h_n$ represent the parts in length and position; also let $\xi_i', \xi_i'', \cdots, \xi_i^{(p)}$, and ξ_i denote any numbers in the part h_i $(i = 1, 2, \cdots, n)$. Then

$$\lim_{n=\infty}\sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)})h_i = \lim_{n=\infty}\sum_{i=1}^{n} F(\xi_i)h_i = \int_a^b F(x)dx.$$

For any difference of the form $a_1a_2\cdots a_p - b_1b_2\cdots b_p$ can be expressed in terms of the differences $a_i - b_i$ by means of the identity

$$a_1a_2\cdots a_p - b_1b_2\cdots b_p = (a_1 - b_1)a_2a_3\cdots a_p + (a_2 - b_2)b_1a_3\cdots a_p$$
$$+ (a_3 - b_3)b_1b_2a_4\cdots a_p + \cdots + (a_p - b_p)b_1b_2\cdots b_{p-1}.$$

* G. A. Bliss, "A Substitute for Duhamel's Theorem," *Annals of Mathematics*, Ser. 2, vol. 16 (1914).

† E. V. Huntington, "On Setting Up a Definite Integral without the Use of Duhamel's Theorem," The American Mathematical Monthly, vol. 24 (1917).

Applying this identity to the difference

$$f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)}) - f_1(\xi_i)f_2(\xi_i)\cdots f_p(\xi_i)$$

and in the coefficient of each difference $f_j(\xi_i^{(j)}) - f_j(\xi_i)$ on the right replacing every factor f by M, the greatest of the maximum absolute values of the several functions $f_j(x)$ in (ab), we have

$$|f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)}) - F(\xi_i)| \leqq |f_1(\xi_i') - f_1(\xi_i)|M^{p-1}$$
$$+ |f_2(\xi_i'') - f_2(\xi_i)|M^{p-1} + \cdots + |f_p(\xi_i^{(p)}) - f_p(\xi_i)|M^{p-1}.$$

But since all the functions $f_j(x)$ are continuous, and therefore uniformly continuous, in (ab), if any positive number ϵ be assigned we can find an integer n' such that for $n > n'$, and $i = 1, 2, \cdots, n$, we shall have

$$|f_1(\xi_i') - f_1(\xi_i)| < \frac{\epsilon}{pM^{p-1}}, \qquad |f_2(\xi_i'') - f_2(\xi_i)| < \frac{\epsilon}{pM^{p-1}}, \qquad \cdots,$$
$$|f_p(\xi_i^{(p)}) - f_p(\xi_i)| < \frac{\epsilon}{pM^{p-1}},$$

and therefore, by the above inequality,

$$|f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)}) - F(\xi_i)| < \epsilon \qquad (i = 1, 2, \cdots, n).$$

From this it immediately follows, since

$$\sum_{i=1}^{n} h_i = b - a,$$

that

$$\left|\sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)})h_i - \sum_{i=1}^{n} F(\xi_i)h_i\right| < \epsilon(b - a),$$

and therefore, since $\epsilon(b - a)$ may be taken as small as we please, that

$$\lim_{n=\infty}\left\{\sum_{i=1}^{n} f_1(\xi_i')f_2(\xi_i'')\cdots f_p(\xi_i^{(p)})h_i - \sum_{i=1}^{n} F(\xi_i)h_i\right\} = 0.$$

Hence since $\lim\limits_{n=\infty}\sum\limits_{i=1}^{n} F(\xi_i)h_i$ exists, and by definition is $\int_a^b F(x)dx$, the theorem is proved.

The theorem and its proof can be immediately extended to the more general intregal

$$\int_V F(x_1, x_2, \cdots, x_m)dV$$

where F denotes a product of p functions f_j all of which are supposed continuous in the closed region V. It is only necessary in the proof just given to replace h_i by the element ΔV_i of V and to interpret the several ξ_i's as symbols for sets of values of $x_1, x_2, \cdots, x_m$ in ΔV_i.

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